Calculus of Complex Valued Functions Part 2: Integral Calculus

Integration of Complex Functions

- In considering extensions of Integral Calculus to the Complex Domain a wide range of new technical apparatus is needed.
- Whereas classical Real integration can be linked to "*area measurement*" this *ceases* to be *true* in the *Complex Plane*.
- Despite the technical intricacies a central result of Complex Integration Theory is of huge significance in important areas of modern Computer Science and, especially, Algorithmics.
- These concern "*counting objects*" and studying the "*average case*" properties of structures.

Some Background

- Suppose $f : C \rightarrow C$ and $p, q \in C$.
- How do we interpret

$$\int_p^q f(z)dz = \int_p^q \mathrm{Re}(f(z))dz + i\int_p^q \mathrm{Im}(f(z))dz$$

- The Complex Numbers are *not ordered*, so (unlike the Reals) we *cannot* think in terms of

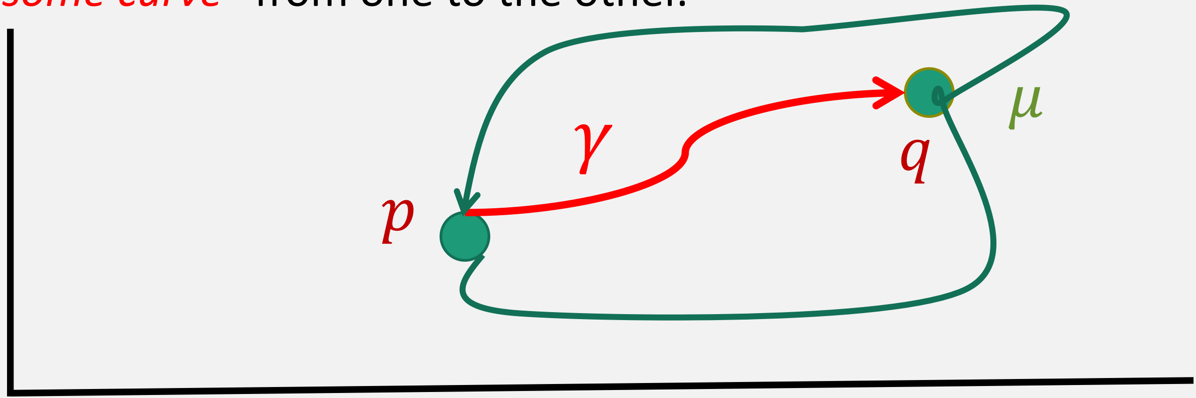
"*some area spanned by $f(z)$ between p and q*"

- We *can*, however,

"*move from (the point) p to (the point) q in the Complex Plane*"

Curves and Contours

- Suppose we have two points – p, q – in the Complex Plane.
- Intuitively, we understand how to "*connect*" these by drawing "*some curve*" from one to the other.



… and "Parameterized" Curves

- We could "*move between points*" by describing the *path* to use (and its *direction*): eg either γ or μ in the example.
- When we do this, the integral concerned is defined as

$$\int_{\gamma} f(z)dz$$

- In the special case of "*closed curves*" (when the start and end coincide) we have so-called "*contour integrals*"

$$\oint_{\mu} f(z)dz$$

Meaning?

- We have an "*interval*" we are interested in: $[\mathrm{Re}(p), \mathrm{Re}(q)]$.
- How do we describe "*moving along a path* γ"?
- By using a function $c : R \to C$ for which
$$c(\mathrm{Re}(p)) = p \;; c(\mathrm{Re}(q)) = q$$
- Let's "*split*" this function into its Real and Imaginary parts
$$x(t) = \mathrm{Re}\big(c(t)\big) \;; y(t) = \mathrm{Im}(c(t))$$
- So that for any t : $\mathrm{Re}(p) \leq t \leq \mathrm{Re}(q)$:
$$c(t) = x(t) + i \cdot y(t)$$
- This is called a "*parameterization of the curve*".

How does this help?

- It can be shown that, when the function $c : R \to C$ parameterizes the curve γ joining p and q then:

$$\int_\gamma f(z)dz \equiv \int_{\mathrm{Re}(p)}^{\mathrm{Re}(q)} f\big(c(t)\big)c'(t)dt$$

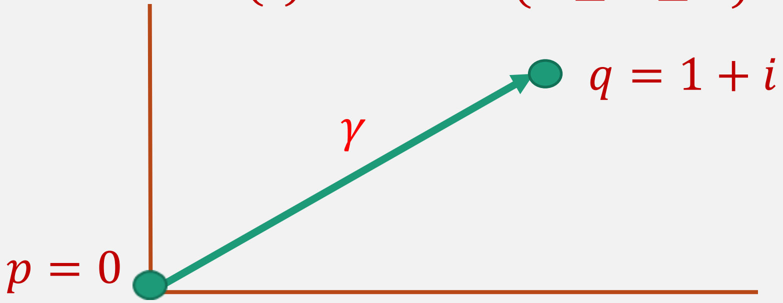
- The *domain* of the right-hand Integral is the *Reals*.
- So we can just use "*standard*" integration by writing:

$$\int_{\mathrm{Re}(p)}^{\mathrm{Re}(q)} \mathrm{Re}\Big(f\big(c(t)\big)c'(t)\Big)\, dt + i\int_{\mathrm{Re}(p)}^{\mathrm{Re}(q)} \mathrm{Im}\Big(f\big(c(t)\big)c'(t)\Big)\, dt$$

An Example: $f(z) = (\mathrm{Re}(z))^2 + i(\mathrm{Im}(z))^2$

- Suppose the interval is $[0,1]$ and we are looking at starting point $p = 0$ and endpoint $q = 1 + i$.
- We could use the parameterized curve $c : R \to C$ with

$$c(t) = t + it \quad (0 \le t \le 1)$$



Example continued

- We want to integrate $f(x+iy) = x^2 + iy^2$ between the points $p = 0$ and $q = 1 + i$.
- We use the curve, γ, with parameterization

$$c(t) = t + it \text{ where } 0 \leq t \leq 1 .$$

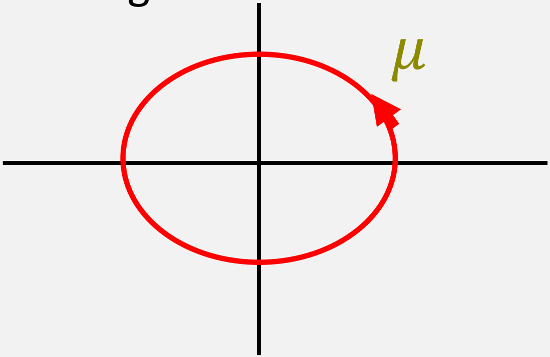
$$c'(t) = 1 + i \; ; \; f\big(c(t)\big) = f(t + it) = t^2 + it^2$$

$$\int_\gamma f(z)dz = \int_0^1 (t^2 + it^2)\,(1 + i)dt$$

$$= (1+i)^2 \int_0^1 t^2 dt = \frac{2i}{3}$$

Another example

- $f(z) = \frac{1}{z}$; $t \in [0,1]$; $c(t) = e^{2\pi i t}$
- This choice of $c(t)$ is the *closed curve* corresponding to a *circle* of radius 1 about the origin:



Example continued further

- We want to integrate $f(x+iy) = \frac{1}{x+iy}$ around the circumference of the unit radius circle, μ
- With parameterization

$$c(t) = e^{2\pi i t} \text{ where } 0 \leq t \leq 1 .$$

$$c'(t) = 2\pi i \cdot e^{2\pi i t} \; ; \; f(c(t)) = f\left(e^{2\pi i t}\right) = e^{-2\pi i t}$$

$$\oint_{\mu} f(z)dz = \int_0^1 f(c(t))c'(t)dt = \int_0^1 \frac{2\pi i \cdot e^{2\pi i t}}{e^{2\pi i t}} dt = 2\pi i$$

How Complex Integrals arise in CS

- The techniques just outlined provide a very basic introduction to elements of Complex Analysis.
- One of the most powerful applications of these ideas arises in the study of counting objects.
- In this case may want to find exact or asymptotic estimates for the "*number of objects of a particular type having size* n".
- Two tools that are often used are the notion of *Generating Function* and a result called the (*Generalized*) *Cauchy Integral Formula*.

A very basic Introduction

- We only have time to give a hint of these ideas and their use.
- Given a *sequence*:

$$\langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle$$

- The sequence may describe "the number of objects of size n" using a_n to denote this ($n \geq 0$). Or deal with other objects.
- For example: "*the number of binary trees with n leaves*", "*the number of permutations of n items that leave* at least one item *in the* same position"; "*the sum of* $^1/_{n^2}$ "
- Being able accurately to estimate such quantities is often crucial in analyzing how an *algorithm behaves on average*.

Generating Functions

- Suppose we write:

$$G(z) = \sum_{n=0}^{\infty} a_n z^n$$

- So the *coefficient* of z^n describes the number of interest.
- How does this help?
- Because we can often *manipulate the sum* (depending on the object of interest) to obtain a "*simple(r) closed form*".

An Example

- Consider the "*binary tree counting*" problem using:

$$B(z) = \sum_{n=0}^{\infty} t_n z^n$$

- A binary tree is *either* a *single root* node ($n = 0$) or a *root* node and *two sub-trees* whose *numbers of leaves added* is n.
- After some manipulation this allows $B(z)$ to be written as:

$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

- If we can find the coefficient of z^n "easily" then we are done.

The Generalized Cauchy Integral Formula

- This is a very deep and remarkable result in Complex Analysis.
- If we want to find a_n from a "*closed form*" (say $g(z)$for $G(z)$ its generating function) we could "*differentiate $g(z)$ n times*" and extract the value we need. *OR* letting $g^{(n)}(z)$ denote this derivative:
- we can use the *Generalized Cauchy Integral Theorem*. Choose a *closed contour*, C, then:

$$g^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{g(\beta)d\beta}{(\beta - z)^{n+1}}$$

- and just evaluate this at $z = 0$.

Complex Integration and its use in CS

- We have (deliberately) skipped over a *significant amount of detail*, not only in discussing *Generating Functions* but in the *full* use of *Cauchy's Integral Formula* and its consequences.
- A sufficient in-depth treatment of *either topic* would need an *entire module* to present.
- Complex analysis is, however, of *great importance* in some *specialized* areas of *algorithmics*: *counting*, finding *good estimates* of "*hard to count*" quantities, *average-case* studies.
- The aim of this section of the module has been just to give a basic sense of what such methods involve.